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A realization of chiral symmetry in Wilsonian RG

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Abstract

The infra-red cutoff used in the Wilsonian RG is often incompatible with standard realization of symmetries. We consider a new realization of the chiral symmetry in a fermionic self-interacting system with a regularization that breaks the symmetry. The cutoff dependent symmetry is characterized by the master equation in the Batalin–Vilkovisky antifield formalism. We also explain how the Polchinski RG flow equation respects symmetries. In particular, we discuss the properties of the Polchinski equation under the cutoff dependent chiral symmetry.

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1. Introduction

The Wilsonian renormalization group provides us with a powerful non-perturbative approach to field theories. Some of its fruitful results have been reported by the speakers at the conference 'Renormalization Group 2005'.

The subject of the present paper is related to the generic conflict between a symmetry and an IR cutoff in the Wilsonian RG approach. In particular, the situation is quite serious for the case of a gauge theory. The local symmetry cannot cope with the presence of the momentum cutoff. Several approaches have been introduced to solve this problem. The fine-tuning approach started with the work by Becchi [1] and was further pursued in [2, 3]. In the approach initiated by Morris and his collaborators [4], they found a way to regularize the pure gauge theory in a gauge invariant manner.

Our viewpoint to the problem is quite different from the works mentioned above. In our earlier papers [5, 6], we have shown that a symmetry is present in the system even after introducing an IR cutoff. It may sound a strong statement. However, let us remember that there is a well-known case of the regularization dependent realization of the chiral symmetry on the lattice. This was done by Lüscher [7] based on the Ginsparg–Wilson [8] relation. Later we will see that our formalism is a kind of generalization to a continuum theory of the Lüscher's symmetry realization.

An important element in our formalism is the Batalin–Vilkovisky antifield formalism [9]. This is a powerful formulation, when one would like to treat any symmetry, local or global. We will see that the presence of a symmetry along the RG flow is summarized as the quantum master equation for the Wilson action.

Earlier we found a way to construct a Wilson action which is invariant under a regularization dependent global symmetry [10]. Once we obtain the action, we would like to see flows of couplings described by the Polchinski equation [11]. As one of the main subjects of this paper, we discuss some important properties of the Polchinski equation under a regularization dependent symmetry. Based on these results, we study the Polchinski equation written for a chiral symmetric theory.

Our formulation is generic and it is, in principle, applicable to a gauge theory [5]. However, in practice, there seems more to understand in solving the master equation and studying the Polchinski equation. So, in this paper, we take a chiral symmetric model for concrete discussions.

The rest of this paper is organized as follows. The next section provides a brief explanation of the antifield formalism and shows the Wilson action satisfies the quantum master equation as the result of the presence of an underlying symmetry. Then we proceed to a discussion of the regularization dependent chiral symmetry. After explaining the important properties of the Polchinski equation, we apply it to a chiral symmetric theory.

2. Wilson action in the antifield formalism

The Batalin–Vilkovisky (BV) antifield formalism [9][4] is a most general quantization method for dealing with local as well as global symmetries. We use this formalism to describe the symmetry in the presence of an IR cutoff. In this section, after briefly describing the antifield formalism, we explain our construction of the Wilson action and its properties which are important for later discussion.

2.1. Antifield formalism

In the antifield formalism, we introduce the antifield ϕ^* for each field variable ϕ and a canonical structure regarding the variables as canonical pairs. The anti-bracket is defined as follows:

$$(F, G)_\phi \equiv \frac{\partial^r F}{\partial\phi^A}\frac{\partial^l G}{\partial\phi^*_A} - \frac{\partial^r F}{\partial\phi^*_A}\frac{\partial^l G}{\partial\phi^A}. \tag{1}$$

Here the superscripts indicate the right and left derivatives, respectively. We use the convention that the repeated index implies the summation.

The quantum action is a functional of fields and antifields, $S[\phi, \phi^*]$. It is constructed in such a manner that it becomes an ordinary action once we remove the antifields. In the antifield formalism, the quantum action works as the generator of a symmetry (λ: anticommuting constant),

$$\phi^A \to \phi^A + (\phi^A, S)_\phi\lambda = \phi^A + \frac{\partial^l S}{\partial\phi^*_A}\lambda, \qquad \phi^*_A \to \phi^*_A + (\phi^*_A, S)_\phi\lambda = \phi^*_A - \frac{\partial^l S}{\partial\phi^A}\lambda. \tag{2}$$

[4] See, for example, [12] for reviews.

Under the change of variables (2), the path integral $\mathcal{D}\phi\exp(-S)$ produces two contributions: (a) the change of the action

$$S \to S + (S, S)_\phi \lambda/2;$$

and (b) the change of the path integral measure

$$\ln \mathcal{D}\phi \to \ln \mathcal{D}\phi + (\Delta_\phi S)\lambda,$$

where

$$\Delta_\phi \equiv (-)^{\epsilon_A+1} \frac{\partial^r}{\partial\phi^A} \frac{\partial^r}{\partial\phi^*_A}. \tag{3}$$

Therefore the path integral is invariant if S obeys the relation

$$\tfrac{1}{2}(S, S)_\phi - \Delta_\phi S = 0, \tag{4}$$

which is the quantum master equation (QME). For later use, we denote the expression on the lhs as $\Sigma[\phi, \phi^*]$ and call it the WT operator; the vanishing of it implies the presence of a symmetry. When the path integral measure is invariant, the QME reduces to the classical master equation (CME),

$$\tfrac{1}{2}(S, S)_\phi = 0. \tag{5}$$

It would be appropriate to mention that the antifield formalism is a sort of generalization of the structure found in an ordinary gauge theory. For a gauge theory, the quantum action could be written in the following form $S[\phi, \phi^*] = S_0[\phi] + \phi^*_A \cdot \delta_B \phi^A$, where $S_0[\phi]$ is an ordinary gauge invariant action and $\delta_B\phi^A$ are the BRS transformations. Thus, the antifields appear as sources for the BRS transformations. It is easy to see that the QME is equivalent to the gauge invariance of the action $S_0[\phi]$, and the field transformations of (2) give us the ordinary BRS transformations.

2.2. *Wilson (average) action*

Let us consider the path integral in the antifield formalism,

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^* \prod_A \delta(\phi^*_A) \exp(-S[\phi, \phi^*]). \tag{6}$$

Here we would like to gradually integrate out the high momentum modes with a smooth IR cutoff. In order to achieve that, we use the method of the average action [13]. Take the Gaussian integral with some function f_k and kernel R^k_{AB} [5]

$$1 = N_k \int \mathcal{D}\Phi\mathcal{D}\Phi^* \prod_A \delta(\Phi^*_A - f_k^{-1}\phi^*_A) \exp\left[-\frac{1}{2}(\Phi^B - f_k\phi^B) R^k_{BC} (\Phi^C - f_k\phi^C)\right]. \tag{7}$$

Inserting (7) into (6) and exchanging the order of integrations, we find the effective action W_k for the fields, Φ and Φ^*,

$$Z = \int \mathcal{D}\Phi\mathcal{D}\Phi^* \prod_A \delta(\Phi^*_A) \exp(-W_k[\Phi, \Phi^*]), \tag{8}$$

[5] The properties of functions f_k and R^k_{AB} will be discussed presently.

where

$$\exp(-W_k[\Phi, \Phi^*]) \equiv N_k \int \mathcal{D}\phi\mathcal{D}\phi^* \prod_A \delta(f_k\Phi^*_A - \phi^*_A) \exp(-S_k[\phi, \Phi, \phi^*]), \tag{9}$$

$$S_k[\phi, \Phi, \phi^*] \equiv S[\phi, \phi^*] + (\Phi - f_k\phi)^A R^k_{AB}(\Phi - f_k\phi)^B/2. \tag{10}$$

In order to realize a gradual integration of high momentum modes, we require some conditions on f_k and R^k_{AB}. The function f_k specifies the IR region,

$$f_k(p) \approx \begin{cases} 0 & \text{for } p^2 > k^2 \\ 1 & \text{for } p^2 < k^2, \end{cases} \tag{11}$$

where k is the IR cutoff. Below the IR cutoff, the kernel R^k_{AB} is to be divergent so that $\Phi^A \sim \phi^A$ in the path integral (9). The high momentum modes of ϕ above the IR cutoff are simply integrated out in (9) since $f_k \approx 0$ in this region.

The fields Φ and Φ^* are the IR fields. The action $W_k[\Phi, \Phi^*]$ is the average action [13] extended to the antifield formalism, which is simply called the Wilson action from now on.

2.3. QMEs for UV and IR fields

Earlier we introduced the WT operator defined by $\Sigma[\phi, \phi^*] \equiv \frac{1}{2}(S, S)_\phi - \Delta_\phi S$. Similarly, we define the WT operator for the action $W_k[\Phi, \Phi^*]$ as $\Sigma_k[\Phi, \Phi^*] \equiv \frac{1}{2}(W_k, W_k)_\Phi - \Delta_\Phi W_k$. Here the anti-bracket and the operator Δ_Φ are defined similarly to (1) and (3) respectively, in terms of the IR fields, Φ and Φ^*.

Let us show that the operator $\Sigma_k[\Phi, \Phi^*]$ may be written as a functional average of the operator $\Sigma[\phi, \phi^*]$. It is easy to obtain

$$\begin{aligned} \Delta_\Phi \exp(-W_k[\Phi, \Phi^*]) &= N_k \int \mathcal{D}\phi\mathcal{D}\phi^* (\Delta_\phi \exp(-S[\phi, \phi^*])) \\ &\quad \times \prod_A \delta(f_k\Phi^*_A - \phi^*_A) \exp\left[-\frac{1}{2}(\Phi^B - f_k\phi^B)R^k_{BC}(\Phi^C - f_k\phi^C)\right] \\ &= N_k \int \mathcal{D}\phi\mathcal{D}\phi^* \prod_A \delta(f_k\Phi^*_A - \phi^*_A) \exp(-S_k[\phi, \Phi, \phi^*])\Sigma[\phi, \phi^*]. \end{aligned}$$

The second equality is due to the relation $\Sigma[\phi, \phi^*] = e^{S[\phi,\phi^*]}\Delta_\phi e^{-S[\phi,\phi^*]}$. Using the similar relation for $\Sigma_k[\Phi, \Phi^*]$, we obtain[6]

$$\Sigma_k[\Phi, \Phi^*] = \langle \Sigma[\phi, \phi^*] \rangle. \tag{12}$$

The functional average of $\Sigma[\phi, \phi^*]$ gives the same quantity defined for the Wilson action. Therefore, if the UV action obeys the QME, then the QME holds for any IR cutoff k. Through a similar argument, one could also show that if the QME holds at some IR cutoff, the same is true for any lower IR cutoff. This can be understood as the presence of the exact symmetry along the RG flow.

3. Polchinski equation is invariant under quantum ‘BRS’ transformation

The master equation is important in order to keep symmetry. Another essential equation is Polchinski’s flow equation. In this section, we describe some important properties of the

[6] Equation (12) is a formal relation and may be obtained without reference to the properties of f_k and R^k_{AB}. By imposing conditions on these functions explained earlier, (12) becomes the relation of WT operators for the UV and IR actions.

Polchinski equation for preserving a symmetry along the RG flow. The results hold generically, not restricted to the chiral symmetry to be considered in the next section.

In the following we show that the change of action under the Polchinski equation may be written as a canonical transformation. In [14], it was shown that the Polchinski equation can be expressed as a change of variables (cf [15]). Our observation here may be regarded as the extension of their result to the antifield formalism.

Under an infinitesimal canonical transformation generated by $G \times \epsilon$, (anti)fields transform as

$$\tilde{\Phi}^A = \Phi^A + (\Phi^A, G)\epsilon, \qquad \tilde{\Phi}^*_A = \Phi^*_A + (\Phi^*_A, G)\epsilon.$$

Accordingly, the action changes as

$$\tilde{W}_k[\tilde{\Phi}^A, \tilde{\Phi}^*_A] - W_k[\Phi^A, \Phi^*_A] = -(G, W_k)\epsilon + \Delta(G\epsilon) = -\delta^k_Q(G\epsilon),$$

where δ^k_Q is the *quantum* BRS transformation defined by

$$\delta^k_Q X \equiv (X, W_k) - \Delta_\Phi X. \tag{13}$$

Note that δ^k_Q is defined in reference to the IR scale k. The quantum BRS transformation satisfies an interesting relation

$$\left(\delta^k_Q\right)^2 X = -(\Sigma_k, X). \tag{14}$$

Therefore the quantum BRS transformation is nilpotent $\left(\delta^k_Q\right)^2 = 0$ if the action satisfies the QME $\Sigma_k = 0$. This is quite important for our discussion.

It was realized in [6] that the canonical transformation with the generator $G_k \times \mathrm{d}k$,

$$G_k = -\frac{1}{2}\Phi^*_A F^{AB}_k \frac{\partial^l W_k}{\partial \Phi^B} - (\partial_k \ln f_k)\Phi^*_A \Phi^A, \tag{15}$$

where $F^{AB}_k \equiv f^2_k \partial_k\left[f^{-2}_k (R^k)^{-1}\right]^{AB}$, induces the change of action under the infinitesimal change of the IR cutoff:

$$-\delta^k_Q G_k = \partial_k W_k + \frac{1}{2}\Phi^*_A F^{AB}_k \frac{\partial^r \Sigma_k}{\partial \Phi^B}. \tag{16}$$

We observe here that the Polchinski equation can be written in the form of the canonical transformation up to the QME. Rewriting the above equation, we have

$$\begin{aligned}\partial_k W_k &= +\frac{1}{2}(-)^A F^{AB}_k \frac{\partial^l \partial^r W_k}{\partial \Phi^B \partial \Phi^A} - \frac{1}{2}\frac{\partial^r W_k}{\partial \Phi^A} F^{AB}_k \frac{\partial^l W_k}{\partial \Phi^B} - \partial_k(\ln f_k)\left[\Phi^A \frac{\partial^l}{\partial \Phi^A} - \Phi^*_A \frac{\partial^l}{\partial \Phi^*_A}\right] W_k \\ &= -\delta^k_Q G_k - \frac{1}{2}\Phi^*_A F^{AB}_k \frac{\partial^r \Sigma_k}{\partial \Phi^B} \approx -\delta^k_Q G_k.\end{aligned} \tag{17}$$

Here $\approx$ may be replaced by the equality when the action satisfies the QME.

Note that (17) may be written as

$$\partial_k \, \mathrm{e}^{-W_k} \approx -\Delta(G\,\mathrm{e}^{-W_k}). \tag{18}$$

Going back to the expression for the partition function, we realize that the rhs of (18) vanishes under the path integration. This may be regarded as another expression of the cutoff independence of the partition function.

It is easy to see from (17) that the derivative of W_k vanishes under δ^k_Q owing to the nilpotency of the quantum BRS transformation. This tells us how the RG flow respects the symmetry.

A comment is in order. In (16), we observe that the flow for a particular regularization may be induced by the corresponding generator in (15). For a different regularization, we have an associated generator for it. This implies that a change of the regularization, or a change of the function F^{AB}, is also induced via a canonical transformation.

4. Chiral symmetry

We take the chiral symmetry to test our formalism. It will be explained that our formalism leads us to a continuum version of the lattice chiral symmetry. In the next subsection, we explain how we construct a Wilson action or how we solve the CME. Later, we discuss the Polchinski equation using the action.

4.1. Construction of Wilson action, solution to classical master equation

Consider a ($d = 4$) chiral invariant UV action $S[\bar{\psi}, \psi]$ for the Dirac fields with the 'BRS' transformation given as

$$\delta\psi(p) = \mathrm{i}c\gamma_5\psi(p), \qquad \delta\bar{\psi}(p) = \mathrm{i}c\bar{\psi}(p)\gamma_5, \tag{19}$$

where, by introducing the constant ghost c, we made the usual chiral transformation in the form of a BRS transformation. The action for a 'block-spin' transformation (cf (10)) is given by

$$S[\bar{\psi}, \psi] + \int_p [\psi^*(-p)\delta\psi(p) + \delta\bar{\psi}(-p)\bar{\psi}^*(p) + (\bar{\Psi} - f_k\bar{\psi})(-p)\alpha(\Psi - f_k\psi)(p)]$$

with $\alpha \equiv \alpha_k(p^2)$,[7] the chiral non-invariant 'mass term' that provides the IR regularization.

Let us consider the free part of $S[\bar{\psi}, \psi]$. The integration over the UV fields produces the free Wilson action,

$$W_k^{(0)} = \int_p [(\bar{\Psi} - \Psi^*\mathrm{i}\gamma_5 c\alpha^{-1})(D - \alpha)(\Psi + \mathrm{i}\gamma_5 c\alpha^{-1}\bar{\Psi}^*) + \bar{\Psi}\alpha\Psi]. \tag{20}$$

Here D is the Dirac operator, $\Psi^* \equiv f_k\psi^*$ and $\bar{\Psi}^* \equiv f_k\bar{\psi}^*$.

From this action (20), we may read off the chiral transformation of the IR fields,

$$\begin{aligned} \delta\Psi &= \left(\Psi, W_k^{(0)}\right) = \mathrm{i}c\gamma_5(1 - (\alpha^{-1}D))\Psi, \\ \delta\bar{\Psi} &= \left(\bar{\Psi}, W_k^{(0)}\right) = \mathrm{i}c\bar{\Psi}(1 - (\alpha^{-1}D))\gamma_5, \end{aligned} \tag{21}$$

which is nothing but the Lüscher's symmetry transformation. It is also easy to find that the CME, $\left(W_k^{(0)}, W_k^{(0)}\right) = 0$, is equivalent to the GW relation

$$D\gamma_5 + \gamma_5 D = 2\alpha^{-1}D\gamma_5 D. \tag{22}$$

In order to simplify the algebraic structure of the chiral symmetry (21), we perform a canonical transformation,

$$\Psi \to \Psi - \mathrm{i}c\gamma_5\alpha^{-1}\bar{\Psi}^*, \qquad \bar{\Psi} \to \bar{\Psi} - \Psi^*\mathrm{i}c\gamma_5\alpha^{-1}, \tag{23}$$

while Ψ^* and $\bar{\Psi}^*$ are left intact. In terms of the new variables, the free action is rewritten as

$$W_k^{(0)} = \int_p [\bar{\Psi}D\Psi + \Psi^*\mathrm{i}c\hat{\gamma}_5\Psi - \bar{\Psi}\mathrm{i}c\gamma_5\bar{\Psi}^*] \tag{24}$$

where $\hat{\gamma}_5 \equiv \gamma_5(1-2(\alpha^{-1}D))$. Correspondingly, the chiral transformation and the GW relation are now expressed as

$$\delta\Psi = \mathrm{i}c\hat{\gamma}_5\Psi, \qquad \delta\bar{\Psi} = \mathrm{i}c\bar{\Psi}\gamma_5 \tag{25}$$

and

$$D\hat{\gamma}_5 + \gamma_5 D = 0. \tag{26}$$

[7] Here we omit the subscript k for simplicity. In later discussion, we do the same for some other quantities.

This is also a well known trick in the case of the lattice chiral symmetry. The fact that $\hat{\gamma}_5^2 = 1$ is found to be important to find chiral invariant interaction terms.

Generically speaking, it is desirable to solve the quantum master equation. However, other than some exceptional cases [16], it seems quite a hard task. So here we consider the classical master equation starting from the free theory: we define the regularization dependent chiral symmetry by using the free theory, then construct interaction terms that respect the symmetry.

The chiral invariant combinations can easily be constructed once we realize that the hatted Dirac field defined below,

$$\hat{\Psi} \equiv (1 - D/\alpha)\Psi,$$

transforms in the standard manner, $\delta\hat{\Psi} = \mathrm{i}c\gamma_5\hat{\Psi}$. All the regularization dependent properties are taken care of by the operator $1 - D/\alpha$. Thus, for example, the chiral and parity invariant 4-fermi operators are given as

$$W_k^{(4)} = \int_{p_1,\ldots,p_4} \left\{ \frac{g_{\mathrm{s}}}{2}[(\bar{\Psi}\hat{\Psi})^2 - (\bar{\Psi}\gamma_5\hat{\Psi})^2] + \frac{g_{\mathrm{v}}}{2}[(\bar{\Psi}\gamma_\mu\hat{\Psi})^2 + (\bar{\Psi}\gamma_5\gamma_\mu\hat{\Psi})^2] \right\}. \tag{27}$$

4.2. Chiral invariance and the Polchinski equation

The Polchinski equation written for our Wilson action is shown below,

$$\begin{aligned}\partial_k W_k = &- \int_q F_{\alpha\beta}(q) \frac{\partial^l \partial^r W_k}{\partial\bar{\Psi}_\beta(-q)\partial\Psi_\alpha(q)} - \int_q \frac{\partial^r W}{\partial\Psi_\alpha(q)} F_{\alpha\beta}(q) \frac{\partial^l W_k}{\partial\bar{\Psi}_\beta(-q)} \\ &- \int_q \partial_k(\ln f_k) \left(\bar{\Psi}_\alpha(-q) \frac{\partial^l W_k}{\partial\bar{\Psi}_\alpha(-q)} + \frac{\partial^r W_k}{\partial\Psi_\alpha(q)} \Psi_\alpha(q) \right) \\ &+ \int_q \partial_k(\ln f_k) \left(\bar{\Psi}^*_\alpha(-q) \frac{\partial^l W_k}{\partial\bar{\Psi}^*_\alpha(-q)} + \frac{\partial^r W_k}{\partial\Psi^*_\alpha(q)} \Psi^*_\alpha(q) \right) + \text{const.}\end{aligned} \tag{28}$$

The function $F_{\alpha\beta}$ is given by the regularization kernel $R_{\alpha\beta}$,

$$F_{\alpha\beta} = f_k^2 \partial_k \left[f_k^{-2} R^{-1} \right]_{\alpha\beta}. \tag{29}$$

In (28), the most important and interesting contributions come from two terms of the first line, while the rest are related to the scaling of the IR fields. The first two terms, often called the trace and dumbbell terms respectively, contain non-trivial functional derivatives of the action. Note that, through the trace term, the 6-fermi interaction terms in the action contribute to the runnings of the 4-fermi couplings.

Now let us consider how the symmetry is respected in the Polchinski equation.

In subsection 4.1, we explained a way to construct a Wilson action for a generic k. This was done based on the representation theory of the 'Lüscher's chiral symmetry' and we obtained the action that satisfies CME, rather than QME. Solving the QME is a far more difficult problem and we restrict ourselves to the study of CME in this paper[8].

On differentiating the CME with respect to k, we obtain the relation, $(W_k, \partial_k W_k) = 0$. This tells us how the symmetry ought to be preserved in the Polchinski equation: a certain combination of the functional derivatives of (28) must be invariant under the symmetry generated by W_k.

[8] There exits a suggestive example: a solution to QME is related to a solution to CME via a canonical transformation [16].

Actually, staring from the CME, we obtain interesting relations. Firstly, we have

$$\left(W_k, \frac{\partial W_k}{\partial \Psi_\alpha(q)}\right) = 0. \tag{30}$$

A similar expression may be derived easily for the derivative with respect to the $\bar{\Psi}_\alpha$. (30) tells us that the dumbbell term of (28) is invariant by itself. It is also easy to see that the scaling terms as a whole are invariant. Using (30), we find the transformation of the sum of scaling terms as

$$-\left(W_k, \bar{\Psi}_\alpha \frac{\partial^l W_k}{\partial \bar{\Psi}_\alpha} + \frac{\partial^r W_k}{\partial \Psi_\alpha}\Psi_\alpha - \bar{\Psi}^*_\alpha \frac{\partial^l W_k}{\partial \bar{\Psi}^*_\alpha} - \frac{\partial^r W_k}{\partial \Psi^*_\alpha}\Psi^*_\alpha\right)$$
$$= \delta\bar{\Psi}_\alpha \frac{\partial^l W_k}{\partial \bar{\Psi}_\alpha} + \frac{\partial^r W_k}{\partial \Psi_\alpha}\delta\Psi_\alpha + \delta\bar{\Psi}^*_\alpha \frac{\partial^l W_k}{\partial \bar{\Psi}^*_\alpha} + \frac{\partial^r W_k}{\partial \Psi^*_\alpha}\delta\Psi^*_\alpha = \delta W_k. \tag{31}$$

The last expression is nothing but the CME, $\delta W_k \equiv (W_k, W_k) = 0$.

We are left with the trace term. The condition of its vanishing is found to be non-trivial. From the CME, we obtain the following relation:

$$\left(W_k, F_{\alpha\beta}\frac{\partial^l \partial^r W_k}{\partial \bar{\Psi}_\beta \partial \Psi_\alpha}\right) + F_{\alpha\beta}\left(\frac{\partial^r W_k}{\partial \bar{\Psi}_\beta}, \frac{\partial^r W_k}{\partial \Psi_\alpha}\right) = 0. \tag{32}$$

Note here that all the relations (30–32) are derived solely from the CME without referring to a particular form of the action.

We may further calculate the second term in (32) if we make some assumptions on the form for the action: (1) the action includes terms at most linear in the antifields; (2) the transformation of a field is linear in the field.

Due to the constant ghost, the first assumption is always fulfilled for a chiral symmetric action. The action constructed in subsection 4.1 satisfies the second assumption as well (cf (21) and (25)). However, the Polchinski equation tends to induce terms nonlinear in fields even for antifield linear terms: when it happens, the transformations of fields become nonlinear. Therefore we may have to relax the second assumption while we search for a truncation scheme.

Under these assumptions, (32) may be rewritten as

$$\left(W_k, F_{\alpha\beta}\frac{\partial^l \partial^r W_k}{\partial \bar{\Psi}_\beta \partial \Psi_\alpha}\right) = -\mathrm{i}c(\hat{\gamma}_5 F + F\gamma_5)_{\alpha\beta}\frac{\partial^l}{\partial \bar{\Psi}_\beta}\frac{\partial^r W_k}{\partial \Psi_\alpha}. \tag{33}$$

The vanishing of the rhs of (33) is the condition for the invariance of the trace term in the Polchinski equation.

Now we would like to see, in concrete examples, how the symmetry could be preserved. We take two distinctive regularizations: one preserves the standard chiral symmetry and the other does not.

First, we consider a chiral symmetry preserving regularization, that is, the regularization kernel $R_{\alpha\beta}$ is chosen to be proportional to $\not{p}$, or $R^{-1} = (\mathrm{i}\not{p})^{-1}\eta(p^2)$ with some function $\eta(p^2)$. Finding a solution to CME is straightforward. The Wilson action is to be written in terms of the standard chiral invariant quantities:

$$\hat{W}_k = \int_p [\bar{\Psi}\mathrm{i}\not{p}\Psi + \Psi^*\mathrm{i}c\gamma_5\Psi - \bar{\Psi}\mathrm{i}c\gamma_5\bar{\Psi}^*] + \hat{W}_k^{(4)} + \hat{W}_k^{(6)}, \tag{34}$$

$$\hat{W}_k^{(4)} \equiv \int_{p_1,\ldots,p_4}\left\{\frac{\hat{g}_s}{2}[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2] + \frac{\hat{g}_v}{2}[(\bar{\Psi}\gamma_\mu\Psi)^2 + (\bar{\Psi}\gamma_5\gamma_\mu\Psi)^2]\right\},$$

$$\hat{W}_k^{(6)} \equiv -\int_p \frac{\partial^r \hat{W}_k^{(4)}}{\partial\Psi}\Delta_H\frac{\partial^l \hat{W}_k^{(4)}}{\partial\bar{\Psi}}, \tag{35}$$

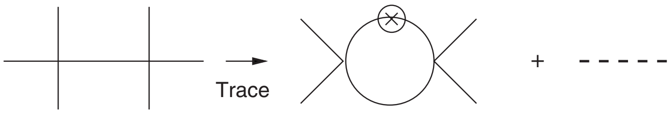

Figure 1. The trace term in (28) out of the 6-fermi interaction (35) contributes to the runnings of 4-fermi couplings. The cross denotes the function $F_{\alpha\beta}$.

where $\Delta_H = (1 - f_k)/\mathrm{i}\not{p}$. We wrote here the terms that contribute to the runnings of 4-fermi couplings. The necessity of the particular type of 6-fermi interactions (35) was shown in [17]. There, via a comparison between the Wilson action and the Legendre effective action, it was shown that, including (35), the former produces the same result as the latter as for the runnings of the 4-fermi couplings. Figure 1 shows how 6-fermi interactions contribute to the trace term in the Polchinski equation. The cross denotes the $F_{\alpha\beta}$ function, which contains the cutoff derivative of the regularization kernel.

The action (34) generates the standard chiral symmetry. It is easily realized that the factor on the rhs of (33) is now simply $\gamma_5 F + F\gamma_5$, which vanishes. In this manner, the Polchinski equation preserves the (standard) chiral symmetry.

Now let us move to the second regularization scheme: we allow the function $R_{\alpha\beta}$ to have a mass-like term. Thus, the inverse of $R_{\alpha\beta}$ takes the following form: $R^{-1} = \eta(\mathrm{i}\not{p})^{-1} + \alpha^{-1}$ with some functions $\eta(p^2)$ and $\alpha(p^2)$. We choose such a function on purpose to break the standard chiral symmetry and see how the regularization dependent symmetry could be realized.

Since the Polchinski equation must respect the symmetry, the rhs of (33) should vanish. Here we consider a particular solution for which the factor in front of the functional derivatives vanishes,

$$\hat{\gamma}_5 F + F\gamma_5 = 0. \tag{36}$$

Note that the condition (36) on $F_{\alpha\beta}$ is nothing but the GW relation if we identify $F_{\alpha\beta}$ with the inverse of the Dirac operator. With an proportionality function κ, let us write[9]

$$F_{\alpha\beta} = f_k^2 \partial_k \left[f_k^{-2} (R^{-1}) \right]_{\alpha\beta} \equiv \kappa (D^{-1})_{\alpha\beta}. \tag{37}$$

At this point, we realize an interesting possibility of satisfying the Polchinski equation as well as the CME: take (37) as the equation that provides D^{-1} from the function $F_{\alpha\beta}$. Using this Dirac operator, we may write the Wilson action in the form of (24) and (27) that is invariant under the regularization dependent symmetry. From our construction, the Polchinski equation is invariant under the transformation generated by the Wilson action.

The action obtained in this way satisfies the CME and is compatible with the Polchinski equation. These are quite non-trivial features. Though further study is necessary, the action may allow us a concrete study of flow of couplings with a symmetry respecting truncation scheme.

5. Summary and discussion

We have shown that the regularization dependent symmetry is present along the RG flow, once it is realized at some IR scale. The associated Ward–Takahashi identities are summarized in the form of the quantum master equation. This fact is simply expressed as (12).

[9] Another possibility of relating $F_{\alpha\beta}$ and D^{-1} is $F_{\alpha\beta} \propto (\hat{\gamma}_5 D^{-1} \gamma_5)_{\alpha\beta}$.

When applied to a chiral symmetric theory, we can encode the Lüscher-like regularization dependent chiral symmetry in the antifield formalism. For the free action, the master equation is nothing but the continuum version of the GW relation. Furthermore, we explained a way to construct a chiral invariant action based on the CME. Having chosen our action satisfying the CME, we investigated the Polchinski equation. From the CME, we studied requirements for the chiral invariant flow. The invariance of the trace term is found to be non-trivial. However, we realized that there is a particular solution to the condition. Based on this observation, we argued that we may construct the action satisfying the CME and consistent with the Polchinski equation.

We are in a position to calculate flows of 4-fermi couplings in our formalism for two kinds of distinctive regularizations, and compare their results to those obtained earlier [18]. The problem is under investigation.

We would like to emphasize that the Polchinski equation has an important property: it is invariant under the quantum BRS transformation. This result is generic, not restricted to the concrete theories we considered here.

Some comments are in order. Naturally, it is desirable to have more general solutions to the QME and then consider the Polchinski equation. However, since it seems quite difficult to solve the QME generically, we restricted our study to solutions of CME in this paper. For a theory without an anomaly to a symmetry of our interest, it could be quite possible that there is a natural choice of field variables for which the action satisfies the CME. If this is the case, the consideration in this paper would be applicable to theories in that category.

We used the Wilson action rather than the Legendre effective action since a symmetry can be treated more naturally. Of course, these two actions carries the same information. Actually, their relation has been reported earlier. In particular, for a gauge theory, the QME for the average action is equivalent to the 'modified WT identity (mWI)' for the Legendre effective action [6]. In an alternative approach [19] with the background field method, the gauge invariance of Legendre effective action as well as the flow equation is discussed by using mWI.

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